

# Wharton

---

**Financial  
Institutions  
Center**

*On Measuring Skewness and Kurtosis  
in Short Rate Distributions:  
The Case of the US Dollar London  
Inter Bank Offer Rates*

by  
**Kabir K. Dutta**  
**David F. Babbel**

**02-25**

The Wharton School  
**University of Pennsylvania**



## The Wharton Financial Institutions Center

The Wharton Financial Institutions Center provides a multi-disciplinary research approach to the problems and opportunities facing the financial services industry in its search for competitive excellence. The Center's research focuses on the issues related to managing risk at the firm level as well as ways to improve productivity and performance.

The Center fosters the development of a community of faculty, visiting scholars and Ph.D. candidates whose research interests complement and support the mission of the Center. The Center works closely with industry executives and practitioners to ensure that its research is informed by the operating realities and competitive demands facing industry participants as they pursue competitive excellence.

Copies of the working papers summarized here are available from the Center. If you would like to learn more about the Center or become a member of our research community, please let us know of your interest.

  
Franklin Allen  
Co-Director

  
Richard J. Herring  
Co-Director

*The Working Paper Series is made possible by a generous  
grant from the Alfred P. Sloan Foundation*

# On Measuring Skewness and Kurtosis in Short Rate Distributions: The Case of the US Dollar London Inter Bank Offer Rates

Kabir K. Dutta<sup>♦</sup>

David F. Babbel<sup>♦</sup>

## Abstract

It has been observed that return distributions in general and interest rates in particular exhibit skewness and kurtosis that are inadequately modeled by the lognormal distribution.

We have modeled the skewness and kurtosis of the short rate using the g-and-h distribution and Generalized Beta Distribution of the Second Kind (GB2) and compare their performance. The g-and-h distribution is a functional transformation of the standard normal distribution and spans a much wider area in the skewness-kurtosis plane than many well-known skewed and leptokurtic distributions including GB2. Researchers have used these distributions in modeling many different asset price and return distributions. However, none of these works have dealt with interest rates of any kind, nor did they compare the fit of their proposed distributions to those of others.

GB2 and g-and-h are both four-parameter distributions and many well-known distributions can be derived as special cases of parameter values from each of these distributions. We observed that the g-and-h distribution exhibited a very high degree of accuracy in modeling the US Dollar 1- and 3-month historical LIBOR rates, and much more than exhibited by GB2.

Key Words: g-and-h, GB2, Lognormal, LIBOR, Skewness, and Kurtosis.

---

<sup>♦</sup> Senior Consultant, NERA

**Contact information:** Kabir.Dutta@NERA.com

National Economic Research Associates, 1166 Avenue of the Americas, 34th Floor, New York, New York 10036

<sup>♦</sup> Special Consultant to NERA, and Professor, The Wharton School, University of Pennsylvania

**Contact information:** babbel@wharton.upenn.edu

3641 Locust Walk, #304, Wharton School, University of Pennsylvania, Philadelphia, PA 19104-6218

## Introduction

Demmel (1999), Babbs and Webber (1998), El-Jahel, Lindberg and Perraudin (1998), and Becker (1991) observed that daily short term interest rates exhibit more skewness and kurtosis than permitted under the assumptions of lognormality. Many different distributions have been proposed to model the skewness and kurtosis often observed in the asset return data. Notable among them are the g-and-h distribution, stable Paretian (e.g., Burr III), generalized beta distribution of second kind (GB2), and extreme value distributions (e.g., Weibull).

We analyze here empirically the shape of the distribution of US dollar 1- and 3-month-LIBOR (London Inter Bank Offer Rate) rates often used to proxy for the short-rate. Knowledge of the distribution of LIBOR of various tenors is an important step for pricing assets contingent on this them, such as US dollar interest rate caps/floors, swaps, and swaptions.

Badrinath and Chatterjee (1988 and 1991), Bookstaber and McDonald (1987), Mills (1995), and McDonald (1997) are some examples where the objectives were to empirically observe the behavior of one particular type of asset return and fit it with an appropriate statistical distribution. However none of these works dealt with interest rates of any kind, nor did they compare the fit of their proposed distributions to that of others.

We study here the skewness and kurtosis of LIBOR rates using the g-and-h distribution and GB2. The g-and-h distribution was introduced by Tukey in 1977 to study asymmetry in income distribution. This distribution, which is a functional transformation of the standard normal, spans a much wider area in the skewness-kurtosis plane than that spanned by many other well-known leptokurtic distributions.

The GB2 was popularized by Richard Bookstaber and James McDonald (1987) and applied to model various asset returns (see also McDonald and Xu, 1995 and McDonald, 1996). Both g-and-h and GB2 are four parameter distributions that capture well skewness and kurtosis. Both perform significantly better than simpler distributions such as lognormal, Burr III, Weibull, and most members of the Pearsonian family of distributions, so our comparison in this paper is restricted to comparing g-and-h with GB2. We will evaluate their accuracy in modeling the skewness and kurtosis of the LIBOR rates by using the goodness of fit test. In a separate paper, we test the efficacy of various assumed underlying distributions, including g-and-h and GB2, in the pricing of interest rate options (see Dutta and Babbel, 2002).

## Short Rate Data

We obtained US Dollar 1- and 3-month LIBOR data from Bridge CRB. These rates are the most heavily traded interest rates. Our analysis period ranged from January 2, 1987 to September 19, 2001. Taking 21 trading days in each month, we analyzed the data in three different (long, medium, and short) lengths of time series: 5 years (1260 days), 1 year (252 days), and 3 months (63 days). On each day we computed volatility, skewness, and kurtosis of LIBOR. These computations are plotted in Figures 1, 2, and 3. We observed frequent and substantial changes in skewness and kurtosis in both rates and across all three estimation durations.<sup>1</sup>

---

<sup>1</sup> In an extended version of this paper, we also examined 6-month LIBOR, and examined all three LIBOR rate series

Taking the natural logarithm of the rates, we computed the Jerque-Bera (JB) statistic for each of the rates and estimation durations. Under the null hypothesis of normality, the JB statistic has an asymptotic  $\chi_2^2$  distribution. The critical value at the 99% significance level is 9.21. Table 1 gives the LIBOR Jerque-Bera statistics. The much higher JB statistic values in all cases indicate that the rates in our sample period did not follow (at a 99% confidence level) a lognormal distribution.

Based on this empirical observation, we anticipated that a leptokurtic distribution with wide coverage in the skewness-kurtosis plane would be a good candidate to model the short rates. The g-and-h distribution has such properties, and so does GB2, albeit to a lesser degree. In the following sections we analyze some of the properties of the g-and-h distribution before we fit it to our data. This is followed by a discussion of GB2 properties, and a fit to our data.

## **g-and-h Distribution**

The g-and-h distribution was introduced by Tukey (1977). Martinez and Iglewicz (1984), Hoaglin (1985), Badrinath and Chatterjee (1988 and 1991), and Mills (1995) also studied the properties of this distribution. Badrinath and Chatterjee, and Mills used the g-and-h distribution to model returns on various equity market indices.

Tukey introduced a family of distributions by transforming the standard normal variable  $Z$  to

$$Y_{g,h}(Z) = (e^{gZ} - 1) \frac{\exp(hZ^2/2)}{g},$$

where  $g$  and  $h$  are any real numbers. By introducing location ( $A$ ) and scale ( $B$ ) parameters, the g-and-h distribution has four parameters in the following form:

$$X_{g,h}(Z) = A + B(e^{gZ} - 1) \frac{\exp(hZ^2/2)}{g} = A + BY_{g,h} \quad (1)$$

When  $h=0$ , the g-and-h distribution reduces to  $X_{g,0}(Z) = A + B \frac{(e^{gZ} - 1)}{g}$ , which is also known as the g-distribution. The  $g$  parameter is responsible for the skewness of the g-and-h distribution. The g-distribution exhibits skewness but no kurtosis.

Similarly when  $g=0$ , the g-and-h distribution reduces to

$$X_{0,h}(Z) = A + BZ \exp(hZ^2/2) = A + BY_{0,h}, \quad (1a)$$

which is also known as the h-distribution. The  $h$  parameter in g-and-h distribution is responsible for its kurtosis. The h-distribution has fat tails (kurtosis) but no skewness.

Before we use the g-and-h distribution to model LIBOR, let us study some of the useful structural properties of the g-and-h distribution, the g-distribution, and the h-distribution.

---

using seven different durations: 3 months, 6 months, 1 year, 18 months, 2 years, 3 years, 4 years, and 5 years. Our results were qualitatively similar to those shown here.

Property 1  $X_{g,h}(Z)$  is a strictly increasing function of  $Z$

Thus the transformation of a standard normal to g-and-h is one to one.

Property 2 If  $A = 0$ , then  $X_{-g,h}(Z) = -X_{g,h}(-Z)$

Property 2 implies that changing the sign of  $g$  changes the direction but not the value of skewness of the g-and-h distribution.

Property 3 For  $h=0$  and  $g \rightarrow 0$ , the g-and-h distribution converges to a normal distribution.

Property 4 The g-distribution is a shifted log-normal distribution

In what follows, unless we need to state the parameters explicitly, we will omit the subscripts representing the parameters  $g$  and  $h$  in the  $X$  and  $Y$  variables.

Next, we will derive Property 5. If  $X_p$  and  $Z_p$  are the  $p^{th}$  percentile of the g-distribution and standard normal distribution respectively, then  $X_p - A = \frac{B}{g} (\exp(gZ_p) - 1)$  (2)

Changing  $p$  to  $1-p$  we have  $X_{1-p} - A = \frac{B}{g} (\exp(gZ_{1-p}) - 1)$  (3)

Observing that  $Z_p = -Z_{1-p}$  and taking the logarithm of the division of (2) by (3) we have

$$g_p = -\left(\frac{1}{Z_p}\right) \ln\left(\frac{X_{1-p} - A}{A - X_p}\right) \quad (4)$$

We also note that  $Z_{0.5} = 0$ . Substituting in (1) we have  $A = X_{0.5}$ . Substituting the value of  $A$  in (4) we have proved:

Property 5 Location parameter  $A$  of the g-and-h distribution is the median of the data set and the  $p^{th}$  percentile of the parameter value  $g$  is given by  $g_p = -\left(\frac{1}{Z_p}\right) \ln\left(\frac{X_{1-p} - X_{0.5}}{X_{0.5} - X_p}\right)$ .

Property 5 implies that by choosing different values of  $p$  one can obtain different estimates of the parameter  $g$ . These estimates of  $g$  for different values of  $p$  reflect the changes in the skewness of the data. The question is: which of these  $g_p$ 's is a good estimate for  $g$ ? Some authors have suggested the use of the median of the  $g_p$ 's. We will develop an estimation procedure for  $g$  later in this chapter using the values of  $g_p$ .

Using equation (1) we have

$$X_p = A + B \left( e^{gZ_p} - 1 \right) \frac{\exp(hZ_p^2/2)}{g} \quad (5)$$

$$X_{1-p} = A + B \left( e^{gZ_{1-p}} - 1 \right) \frac{\exp(hZ_{1-p}^2/2)}{g} \quad (6)$$

Using  $Z_p = -Z_{1-p}$  and subtracting (6) from (5) we obtain:

Property 6 For a given value of  $g$ , the value of  $h$  in the g-and-h distribution is given by

$$\frac{g(X_p - X_{1-p})}{(e^{gZ_p} - e^{-gZ_p})} = B \exp(hZ_p^2/2) \quad (7)$$

Or,

$$\ln \frac{g(X_p - X_{1-p})}{(e^{gZ_p} - e^{-gZ_p})} = \ln(B) + h(Z_p^2/2) \quad (8)$$

Thus we can obtain the value of  $h$  from the linear regression of  $\ln \frac{g(X_p - X_{1-p})}{(e^{gZ_p} - e^{-gZ_p})}$  on  $Z_p^2/2$ . The coefficient of the regression gives the value of  $h$  and the  $\exp(\text{intercept})$  gives the value of  $B$ . We thus obtain the values of  $h$  and  $B$  conditional on the value of  $g$ . The left hand side of equation (7) is also known as the corrected full spread (CFS). Here we have a single estimate for  $h$ , the parameter responsible for the kurtosis of the g-and-h distribution. Later we will explore other ways of estimating the value of  $h$  conditional on  $g$ . If we use two halves of the data separately, then we can obtain two different estimates for  $h$  and thus two different estimates of the kurtosis for two tails of the distribution. The upper and lower half spreads (UHS and LHS) are given by the following two equations:

$$\text{UHS} = \frac{g(X_{1-p} - X_{0.5})}{(e^{-gZ_p} - 1)} \quad (9)$$

$$\text{LHS} = \frac{g(X_{0.5} - X_p)}{(1 - e^{gZ_p})} \quad (10)$$

Replacing CFS in equation (7) by UHS and LHS we obtain  $h$  for the upper and lower half of the data.

Martinez and Iglewicz (1984) have shown that the g-and-h distribution covers most of the Pearsonian family of distributions up to an adequate approximation and can also generate a variety of other types of distributions. They show how more than more than twelve distributions can be generated from the g-and-h distribution by choosing the values of  $A$ ,  $B$ ,  $g$ , and  $h$ .<sup>2</sup> We saw earlier that normal and lognormal distributions are two very important special cases of the g-and-h distribution. This property of the g-and-h distribution makes it a very general leptokurtic distribution for modeling asset returns.

In order to study the skewness and kurtosis of a distribution we need to evaluate its moments. Martinez and Iglewicz (1984), and Hoaglin (1985) have shown that calculations for the moments of the g-and-h distribution are straightforward, although it becomes tedious as the order of the moment increases.

---

<sup>2</sup> In addition to normal and lognormal distributions, they showed how to generate uniform, student-t, exponential, double exponential, cauchy, beta, gamma, Weibull, Chi-square, and logistic distributions. We can also get Burr III as a combination of the beta and gamma distributions.

## Estimation of Moments

To compute the moments of the g-and-h distribution, we first compute the moments of the g-distribution (when  $h = 0$ ) and h-distribution (when  $g = 0$ ). In Property 4 we noted that the g-distribution is a horizontally shifted lognormal distribution. Therefore, the moments of the g-distribution are similar to those of the lognormal distribution with adjustment for the shift. The h-distribution is a symmetric distribution (because  $g = 0$ ). As a result, all odd-order moments of an h-distribution are zero.

From equation (1) we have  $X^n = (A + BY)^n = \sum_{i=0}^n \binom{n}{i} A^{n-i} B^i Y^i$ . Therefore,

$$E[X^n] = \sum_{i=0}^n \binom{n}{i} A^{n-i} B^i E(Y^i) \quad (11)$$

When  $g$  is not equal to zero

$Y^i = \frac{\exp(ihZ^2/2) \sum_{r=0}^i (-1)^r \binom{i}{r} \exp((i-r)gZ)}{g^i}$ , where  $Z$  is a standard normal variable. Therefore,

$$E(Y^i) = \frac{\sum_{r=0}^i (-1)^r \binom{i}{r} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(1-ih)Z^2 + (i-r)gZ\right) dz}{\sqrt{2\pi} g^i} \quad (12)$$

Completing the squares and rearranging the terms we have

$$E(Y^i) = \frac{\sum_{r=0}^i (-1)^r \binom{i}{r} \exp[(i-r)^2 g^2 / 2(1-ih)]}{(1-ih)^{i/2} g^i} \quad (13)$$

Substituting (13) in (11), we have

$$E(X^n) = \sum_{i=0}^n \binom{n}{i} A^{n-i} B^i \frac{\sum_{r=0}^i (-1)^r \binom{i}{r} \exp[(i-r)^2 g^2 / 2(1-ih)]}{(1-ih)^{i/2} g^i} \quad (14)$$

When  $g = 0$ , then  $Y = Z \exp(hZ^2/2)$  and  $Y^i = Z^i \exp(ihZ^2/2)$ . Therefore,

$E(Y^i) = \frac{\int_{-\infty}^{\infty} Z^i \exp\left(\frac{-1}{2}(1-ih)Z^2\right) dz}{\sqrt{2\pi}}$ . Again completing the squares and rearranging the terms

we have  $E(Y^i) = \frac{i!(1-ih)^{-i+i/2}}{2^{i/2}(i/2)!}$  when  $i$  is even. We have noted earlier that  $E(Y^i) = 0$  when  $i$  is odd. Substituting in (11) we get

$$E(X^n) = \sum_{i=0}^n \binom{n}{i} A^{n-i} B^i f(i) \frac{i!(1-ih)^{-(i+1/2)}}{2^{i/2}(i/2)!} \quad (15)$$

where  $f(i)=0$  when  $i$  is odd and equal to 1 when  $i$  is even.

Using the moments we can compute the skewness and kurtosis of the g-and-h distribution, the computation of which can sometimes be algebraically tedious. Without making the actual computation for skewness and kurtosis we can make some observations about them with the help of the moments calculated in steps (14) and (15).

The g-and-h distribution can have a wide variety of tail behavior. This gives great flexibility in fitting the actual data. Martinez and Iglewicz (1984), and Hoaglin (1985) compare the skewness and kurtosis of the g-and-h distribution with the distributions belonging to the Pearsonian family. The Pearsonian family of distributions can be broadly classified into seven different categories and all have finite first four moments. Martinez and Iglewicz (1984) observed that the g-and-h distribution covered a much wider area in the skewness-kurtosis plane than do the distributions in the Pearsonian family.

## Modeling the LIBORs with g-and-h distribution

We observed earlier that LIBOR rates exhibit skewness and kurtosis beyond that permitted under the assumption of lognormality. The Jerque-Bera statistic indicated that LIBOR rates are not even closely approximated by a lognormal distribution. In this section we fit the g-and-h distribution to LIBOR rates and perform the goodness-of-fit tests. In earlier sections we noted the flexibility of the g-and-h distribution and how it covers a wide range in the skewness-kurtosis plane. Therefore, it is natural to expect that whatever distribution LIBOR rates follow, it can be approximated by a distribution from the g-and-h family. The estimation we will follow here is the method of quantiles introduced by Hoaglin (1985). Alternative methods that could be used are the maximum likelihood and method of moments. Both of these alternative methods will limit the use of the structural richness of the g-and-h distribution. Also these methods can be computationally tedious. The method based on quantiles uses the structural simplicity of the g-and-h distribution and hence is computationally easy. Furthermore, as Hoaglin (1985) noted, the method of quantile uses the tail behavior of the data and not just the centralized summary information of the data used by the method of moments or the maximum likelihood method.

On any day and for either 1-month or 3-month LIBOR rates, our estimation periods each have 63, 252, or 1260 data points. We chose the percentiles from 0.5 to .00007 at a geometric progression with a total of 16 points. Using Property 5 we estimated  $g_p$  for different values of  $p$  (percentile). We plotted  $g_p$  against  $Z_p^2$  and looked for the curvature in the plot. We observed a nonlinear relationship in most of the cases. Therefore, we used the following polynomial to estimate  $g$ :

$$g(Z) = k_0 + k_1 Z^2 + k_2 Z^4 + k_3 Z^6 \quad (16)$$

Increasing the term beyond  $Z^6$  in (16) did not change the value of  $g$  significantly. If the  $R^2$  value of the regression in (16) was at least 95%, we estimated the  $g$  using (16); otherwise, we took the median of the values of  $g_p$ .

Using these values of  $g$ , equation (8), UHS from (9) and LHS from (10),  $h$  was estimated.

When there is a significant difference between the two values for  $h$ , we take the average of the two; otherwise we take the value from UHS. Had we instead used two different values of  $h$  it would have entailed fitting two g-and-h distributions – one for the lower half and the other for the upper half. However, this runs the risk of data over fitting. In order to avoid this and to fit only one g-and-h distribution on any given day, we chose to take the average of the two. In most cases, we observed a very insignificant difference. Some authors have proposed the use of a functional relationship similar to equation (16) for estimating  $h$ . We did not observe any significant difference in estimating values of  $h$  between the two approaches. Using Property 5 and equation (8) we also obtain an estimation for  $A$  and  $B$ . Thus we have fitted the g-and-h distribution to short rates on each day for the previous 63, 252, or 1260 days from 1987 to 2001.

In order to test the goodness of fit, we used equation (1) to compute the percentile values based on the g-and-h distribution and called it  $E_p$ . We compared it with the observed value  $O_p$  using the following equation:

$$\chi^2_\alpha = \sum_{i=1}^{16} \frac{(O_i - E_i)^2}{E_i}, \text{ where } \chi^2_\alpha \text{ is a chi-square distribution with } \alpha \text{ (equal to 11) degrees of}$$

freedom. For each trading day between January 2, 1987 and September 19, 2001 and using three different lengths of time series we observed an extremely good fit for the 3-month and 1-year estimation periods (at a 99% confidence level with 24.75 as the critical point), and a good fit for the 5-year estimation period, even in the tails of the distribution. Table 1 gives the summary statistics of the percentage error in each case.

## Generalized Beta Distribution of Second Kind (GB2)

Generalized Beta Distribution of the Second Kind (GB2), like the g-and-h distribution, can accommodate a wide variety of tail-thickness and permits skewness as well. Bookstaber and McDonald (1987), McDonald (1991 and 1996), and McDonald and Xu (1995) have analyzed the properties and applications of the GB2 distribution in detail. Bookstaber and McDonald (1987), and McDonald (1996) have explored the possibility of modeling asset returns using GB2. The GB2 distribution is defined as:

$$GB2(y; a, b, p, q) = \frac{|a| y^{ap-1}}{b^{ap} B(p, q) [1 + (y/b)^a]^{p+q}} \text{ when } y > 0, \quad (17)$$

$$= 0 \text{ otherwise}$$

Here,  $B(p, q)$  is a Beta function. Like the g-and-h, GB2 is a four parameter distribution. Some of the useful properties of GB2 are summarized below.

The cumulative distribution function of GB2 is given by<sup>3</sup>

$$X(y; a, b, p, q) = z^p {}_1F_2[p, 1 - q, 1 + p; z] / pB(p, q) \quad (17a)$$

where  $z = (y/b)^a / (1 + (y/b)^a)$  and  ${}_1F_2[a, b, c, d]$  is a hypergeometric function.<sup>4</sup>

---

<sup>3</sup> For derivation see McDonald and Xu (1995) and McDonald (1996).

<sup>4</sup>Hypergeometric is a special function. Reference for such functions will be Abramowitz and Stegun (1972).

Property 7 Parameter  $a$  is the location parameter and determines how quickly the tails of the distribution approach the X-axis.

Therefore, a large value of parameter  $a$  implies a sharp peakedness in the distribution. The mean of the GB2 distribution which we will derive in the next section along with other moments, is  $bB(p+1/a, q-1/a)/B(p, q)$ . Parameter  $b$  is the scale parameter. By inspection of the mean we observe

Property 8 For large values of parameter  $a$ , parameter  $b \rightarrow$  mean. Also, doubling of  $b$  will move the mean 100% to the right.

Parameter  $b$  affects the height of the density as well. Parameter  $q$  determines the kurtosis of the distribution. The product  $aq$  directly affects the kurtosis of the distribution. We will see later in this section that moments of GB2  $\geq aq$  do not exist. Parameters  $p$  and  $q$  together determine the skewness of the distribution.

Like g-and-h, GB2 is very flexible and can be shown to include many well-known distributions as a special and or a limiting case. Among the distributions that can be derived as a special case of GB2 are the lognormal ( $a \rightarrow 0, q \rightarrow \infty$ ), the chi-square ( $a=1, q \rightarrow \infty$ ), and the exponential ( $a=1, p=1, q \rightarrow \infty$ ). Bookstaber and McDonald (1987) give a list of other distributions that can be derived from GB2 as a special case of the parameter values. McDonald (1996) derived the following differential equation for the density function  $f(y)$  of GB2:

$$\psi(y) = \frac{d \ln f(y)}{dy} = \frac{ap-1-(aq+1)(y/b)^a}{y(1+(y/b)^a)} \quad (18)$$

and has shown that GB2 neither includes nor is included as a special case in the Pearsonian family whose density is given as a solution to the following differential equation:

$$\psi(s) = \frac{d \ln f(s)}{ds} = \frac{s-a}{b+cs+es^2} \text{ where } a, b, c, \text{ and } e \text{ are real constants.} \quad (19)$$

This is a very important difference between GB2 and the g-and-h distribution. As we noted earlier, the g-and-h distribution does include many distributions of the Pearsonian family.

Since we have not expressed the GB2 distribution as a simple functional transformation of a known distribution (like normal in the case of g-and-h), it would be very difficult to use the method of quantiles to fit the GB2 distribution to LIBOR rates. Therefore, economists and statisticians have used the method of moments to fit this distribution, and we do the same here.

## Estimation of Moments

Since McDonald and Xu (1995) have shown the derivation of the moments for GB2, we will not derive them here again. Instead we will make some observations concerning the moments of GB2 that will be useful for our application. First we state omitting the proof<sup>5</sup> that

Property 10 The  $h$ th-order moment (about the origin) of GB2 distribution is given by

---

<sup>5</sup> For proof see McDonald and Xu (1995).

$$E(Y^h) = \frac{b^h B(p+h/a, q-h/a)}{B(p, q)} \quad (20)$$

When  $h$  is equal to 1 we get the mean :  $E(Y) = \frac{bB(p+1/a, q-1/a)}{B(p, q)} \quad (21)$

From (20) we make the following important observation:

Property 11 No moments of order  $\geq aq$  will exist.

This property explains our earlier observation that  $aq$  has a direct effect on the kurtosis of the distribution. Also, from the functional form of the second parameter of the Beta function in the numerator of (20) we conclude the following:

Property 12 For the GB2 distribution to be of finite variance,  $aq$  must be strictly greater than 2 and to be of finite kurtosis  $aq$  must be strictly greater than 4.

The above property implies that if  $aq > 4$  then we have a finite mean, variance, skewness and kurtosis. The variance of any distribution (when it exists) is given by  $E(Y^2) - E(Y)^2$ . From (20) and (21) we can see that when  $a \rightarrow \infty$  the variance of the distribution tends toward zero. Earlier we observed that when  $a \rightarrow \infty$  the mean of the distribution tends toward  $b$ . With these two observations together we have the following:

Property 13 For large values of  $a$  the probability mass of the GB2 density concentrates near  $b$ .

Having observed some of the important properties of the GB2 distribution and its moments, we now proceed to model LIBOR rates using the GB2 distribution.

## Modeling the 3-month-LIBOR with GB2 distribution

Since the GB2 distribution is a leptokurtic distribution with wide range of skewness and kurtosis, we would like to test how effectively it can model the LIBOR rates. We have seen earlier that we could model effectively the LIBOR rates with the g-and-h distribution.

The estimation we will follow here is the method of moments. Since there are four parameters to be estimated we need moments up to order four. The moments we used were moments about the origin and equated with the moments computed using the data of the previous 63, 252, or 1260 trading days on each day of our analysis period. The four simultaneous equations that were solved to estimate parameters on each day are:

$$\frac{b^h B(p+h/a, q-h/a)}{B(p, q)} = m_h, \text{ where } h \text{ varies from 1 to 4.} \quad (22)$$

The  $m_h$ 's are the first four moments (about the origin) computed with the data. The system of equations had a solution with accuracy of the order 0.0001 in all cases.

In order to test the goodness-of-fit, we use equation (17a) to compute the percentile values based on the parameters estimated in step (22). We kept the same 16 percentile points we used for the g-and-h distribution. As before, we call the estimated value  $E_p$  and then compare it with the observed value  $O_p$  using the following equation:

$$\chi^2_\alpha = \sum_{i=1}^{16} \frac{(O_i - E_i)^2}{E_i}, \text{ where } \chi^2_\alpha \text{ is a chi-square distribution with } \alpha \text{ (equal to 11) degrees of free-}$$

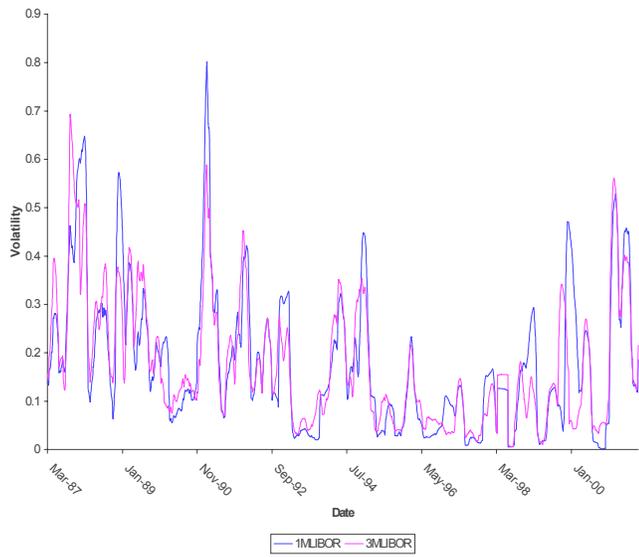
dom. For some of the trading days between January 2, 1987, and September 19, 2001 we did not observe a good fit. The test statistics in general were higher than the test statistics we observed in the case of the g-and-h distribution. We also observed that the fit in the tail of the distribution was not as good as it was with the g-and-h distribution. One possible reason for this could be that the GB2 distribution does not cover as wide an area in the skewness-kurtosis plane as does the g-and-h distribution. Also, as noted earlier under several properties, the GB2 distribution is very sensitive to the value of its parameters. Unlike the g-and-h distribution, the skewness and kurtosis in the GB2 distribution is determined by a combination of parameters. Therefore a good fit with respect to one moment may not necessarily mean a good fit with respect to the other moments. In this regard the g-and-h distribution is more flexible than GB2. Table 1 gives the summary statistics of the percentage error in each case.

## Conclusion

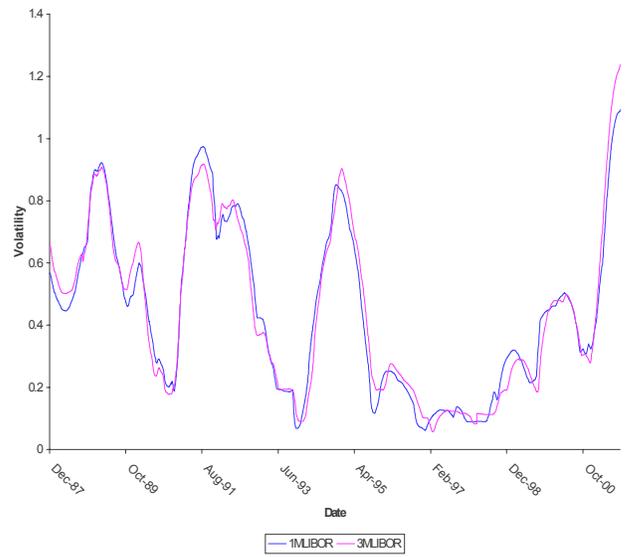
From our empirical analysis of the short rates it is quite obvious that LIBOR did not conform to a lognormal distribution. In addition, we observed a great amount of volatility and changes (both in terms of direction and quantity) in the skewness and kurtosis of the data. We explored the properties of g-and-h here, which has a great deal of flexibility with respect to skewness and kurtosis. Based on our analysis we modeled LIBOR rates with the g-and-h and GB2 distributions. We observed a very high accuracy in modeling the distribution of the LIBORs. The structural simplicity of g-and-h made it computationally much easier to estimate its parameters than those of GB2.

There may be other distributions useful in modeling the historical LIBORs. The GB2 distribution is extremely sensitive to its parameters, so much so that a very small change in the parameter value can drastically alter the value of the skewness and kurtosis. Therefore, computationally if we cannot obtain a very high accuracy in estimating the parameters of GB2, we may be fitting a completely different GB2 than the data actually implied. On the other hand, the g-and-h distribution lends itself to the quantile method of estimation. One distinct advantage of this method is that we could model the tail behavior of the data easily. On the other hand, the quantile method when applied to GB2 distribution can be computationally intractable. Table 1 shows g-and-h is probably a better choice over GB2 in modeling LIBOR data. In our worst measuring intervals, which were 5 years, we found there were about 7% of the cases that did not fall in the 99% confidence interval in the case of g-and-h, but it was about 45% for GB2.

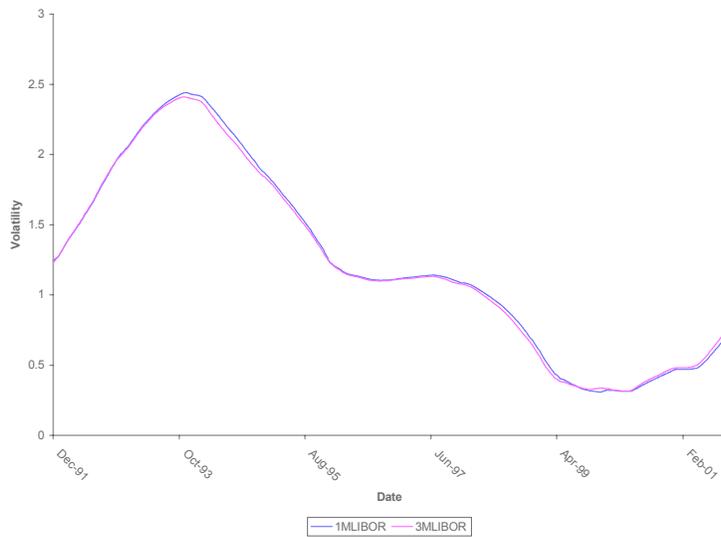
The accuracy we obtained here in modeling the LIBOR rates using the g-and-h distribution is significant enough to conclude that we can effectively use the g-and-h distribution to model the skewness and kurtosis of short rates in general and LIBOR rates in particular. However, we are unable to determine the extent to which the superiority of g-and-h to GB2 is because we used different methods for estimations. Based on the structure of the g-and-h distribution, the quantile method of estimation was the best choice, whereas for GB2 the quantile method is computationally cumbersome and unstable. Therefore, the method of moments was used for GB2. As we can see from the expression of the moments for g-and-h, the method of moments would have been very tedious for estimating the parameters of g-and-h. Therefore, when viewed as a package of a distribution joined with an estimation method, we can say that g-and-h is better than GB2 in this application.



(a)

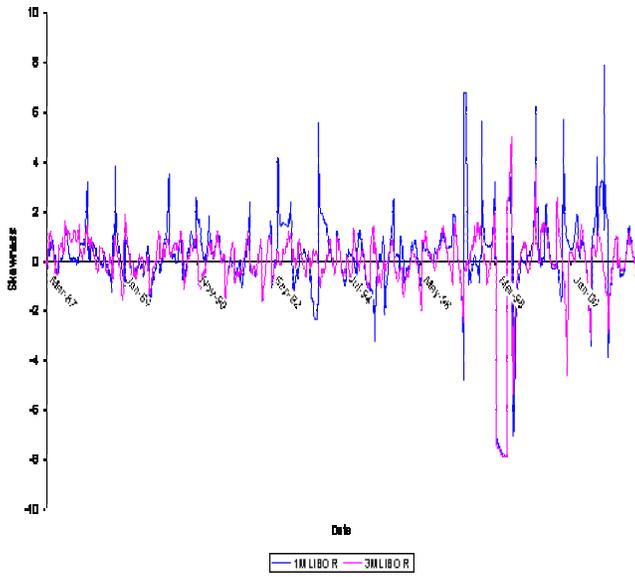


(b)

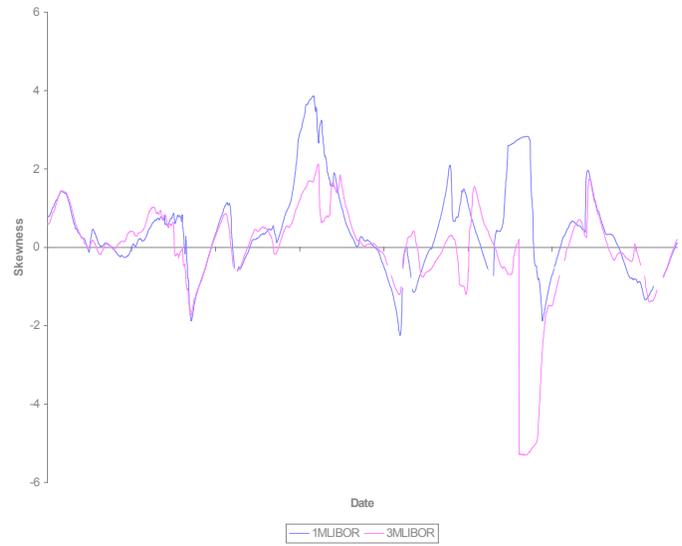


(c)

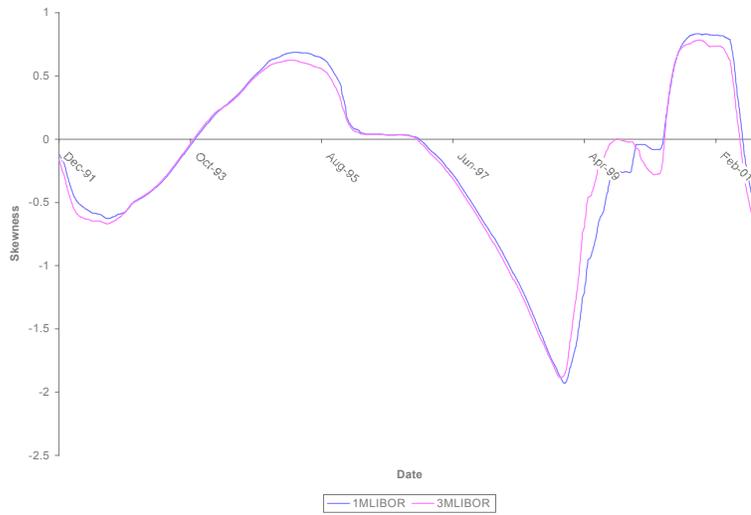
**Figure 1 – (a) 3-month, (b) 1-year, and (c) 5-year volatility of the 1 and 3-month USD LIBOR from January 2, 1987 to September 19, 2001**



(a)

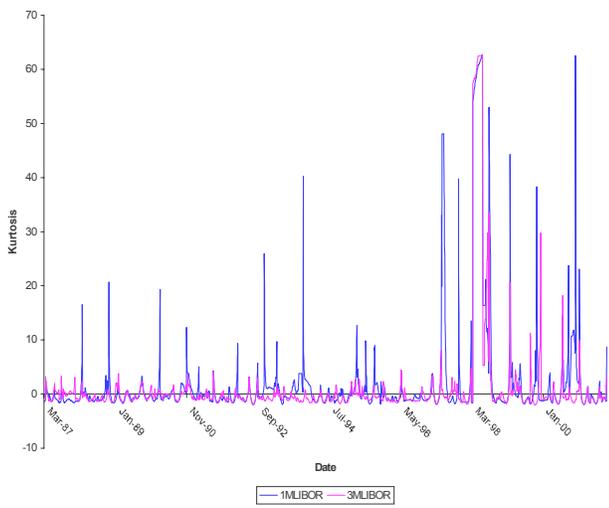


(b)

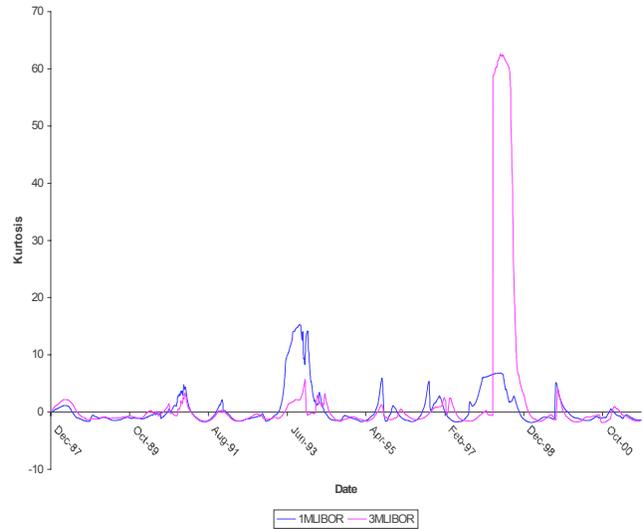


(c)

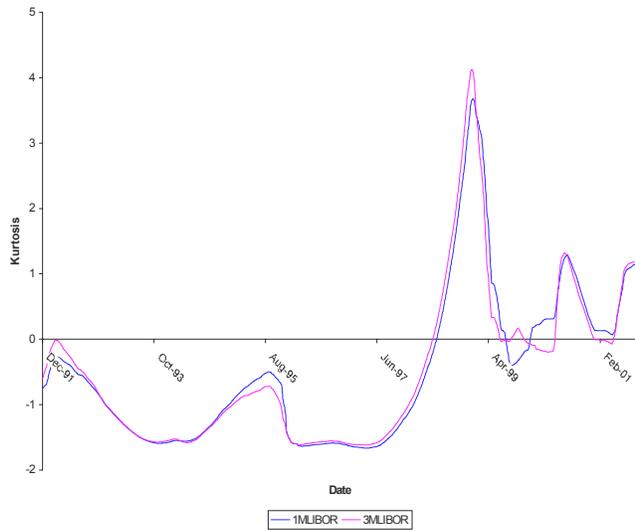
**Figure 2 – (a) 3-month, (b) 1-yr, and (c) 5-yr skewness of 1 and 3-month USD LIBOR from January 2, 1987 to September 19, 2001**



(a)



(b)



(c)

**Figure 3 – (a) 3-month, (b) 1-yr, and (c) 5-yr kurtosis of the 3-month USD LIBOR from September 27, 2000 to September 19, 2001**

**Table 1**

The following presents the summary statistics for the goodness of fit of the g-and-h and GB2 distributions to 1-month and 3-month-LIBOR data. The estimates were made for 63, 252, and 1260 days of the LIBOR data on from January 2, 1987 to September 19, 2001. The table also presents the summary of the Jerque-Bera statistic for the LIBOR data. The critical points of Jerque-Bera statistic and goodness of fit are 9.21 and 24.75 respectively.

		Jerque-Bera Statistics			g-and-h Distribution			GB2 Distribution		
		3M	1Yr	5Yr	3M	1Yr	5Yr	3M	1Yr	5Yr
<b>1MLIBOR</b>	<i>Mean</i>	258.76	612.67	827.90	4.24	1.45	36.78	6.31	17.07	60.02
	<i>Median</i>	42.14	177.58	934.22	0.01	0.16	1.54	6.51	11.48	21.91
	<i>25 Percentile</i>	32.91	146.12	554.05	0.00	0.02	0.18	1.18	4.26	10.54
	<i>75 Percentile</i>	52.02	212.94	1104.77	0.16	0.85	7.50	16.87	23.93	40.65
	<i>95 Percentile</i>	257.28	537.83	1134.90	4.23	4.78	31.60	41.55	53.13	96.00
	<i>% above 9.21 (for Jerque Bera)</i>	100%	100%	100%						
	<i>% above 24.75 (for g-and-h and GB2)</i>				1.72%	1.09%	7.23%	19.31%	25.73%	45.53%
<b>3MLIBOR</b>	<i>Mean</i>	214.59	2143.46	826.57	1.23	2.05	6.15	8.58	20.06	30.40
	<i>Median</i>	39.72	163.43	954.13	0.01	0.17	0.68	6.19	13.50	21.83
	<i>25 Percentile</i>	31.46	127.33	540.40	0.00	0.02	0.13	1.18	5.50	10.05
	<i>75 Percentile</i>	48.55	195.57	1092.89	0.10	0.77	6.89	14.81	27.30	42.14
	<i>95 Percentile</i>	62.03	242.53	1125.10	1.28	7.87	41.21	34.89	60.52	107.95
	<i>% above 9.21 (for Jerque Bera)</i>	100%	100%	100%						
	<i>% above 24.75 (for g-and-h and GB2)</i>				0.40%	1.40%	6.94%	13.37%	29.39%	45.69%

## Bibliography

- Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions*. Washington D.C., National Bureau of Standards, 1972.
- Babbs S. H., and N. J. Webber, "Term Structure Modeling Under Alternative Official Regime." in *Mathematics of Derivative Securities*. Eds. Dempster and Pliska. Cambridge, UK: Cambridge University Press, 1998.
- Badrinath, S. G. and S. Chatterjee, "On Measuring Skewness and Elongation in Common Stock Return Distributions: The Case of the Market Index." *Journal of Business*, Vol. 61, No. 4, 1988.
- Badrinath, S. G. and S. Chatterjee, "A Data-Analytic Look at Skewness and Elongation in Common-Stock-Return Distributions." *Journal of Business & Economic Statistics*, Vol. 9, No. 2, 1991.
- Becker, D. N., *Statistical Tests of The Lognormal Distribution As A Basis For Interest Rate Changes*. Schaumburg, IL: Society of Actuaries, 1991.
- Bookstaber R. M. and J. B. McDonald, "A General Distribution For Describing Security Price Returns." *Journal of Business*, Vol. 60, No. 3, 1987.
- Chen, L., *Interest Rate Dynamics, Derivative Pricing, and Risk Management*. New York, NY: Springer-Verlag, 1996.
- Demmel, R., *Fiscal Policy, Public Debt and the Term Structure of Interest Rates*. Berlin: Springer-Verlag, 1999.
- Dutta, K. K., *Leptokurtic Distributions and Tests of Distributional Assumptions in Extracting Probabilistic Information from Interest Rate Options*. Philadelphia, PA, USA: Ph.D. Dissertation, University of Pennsylvania, 2002.
- Dutta, K. K. and D. F. Babbel, "Extracting Probabilistic Information from the Prices of Interest Rate Options: Tests of Distributional Assumptions." Working Paper, National Economic Research Associates (NERA), May 2002.
- El-Jahel, L., H. Lindberg and W. Perraudin, "Interest Rate Distribution, Yield curve Modeling, and Monetary Policy." In *Mathematics of Derivative Securities*. Eds. Dempster and Pliska. Cambridge, UK: Cambridge University Press, 1998.
- Hoaglin, D. C., "Using Quantiles to Study Shape." Chapter 10 in *Exploring Data Tables Trends, and Shapes*. Ed. Hoaglin, Mosteller, and Tukey. New York, NY: John Willey, 1985.
- Hoaglin, D. C., "Summarizing Shape Numerically: The g-and-h Distributions." Chapter 11 in *Exploring Data Tables Trends, and Shapes*. Eds. Hoaglin, Mosteller, and Tukey. New York, NY: John Willey, 1985.
- Martinez, J. and B. Iglewicz, "Some Properties of the Tukey g and h Family Distributions." *Communications in Statistics - Theory Meth.*, 1984.
- McDonald, J. B. "Probability Distributions for Financial Models." *Statistical Methods in Finance*, Vol. 14 of *Handbook of Statistics*, Eds. Madalla and Rao. Amsterrdam: Elsevier, 1996.
- McDonald, J. B. and Y. J. Xu, "A Generalization of the Beta Distribution with Applications." *Journal of Econometrics*, Vol. 66, 1995.

Mills, T. C., "Modelling Skewness and Kurtosis in the London Stock Exchange FT-SE Index Return Distributions." *The Statistician*, Vol. 44, No. 3, 1995.

Tukey, J. W., *Exploratory Data Analysis*. Reading, MA: Addison-Wesley, 1977.