A Model of Ordered Bargaining with Applications

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Abstract

We present a model of ordered bargaining between one buyer and several sellers, with Nash bargaining at each stage. We first show that the model has the property that the buyer's payoff equals the expected utility of a weighted sum of independent Bernoulli random variables. We then exploit this property to uncover further properties and implications of the model using standard results from expected utility and probability theory. We derive some potential implications for merger analysis and we outline an application of the model to the allocation of pork barrel spending.
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1 Introduction

Business and political decision-making protocols often involve multi-party negotiations between a "lead negotiator" and several business partners or political allies. For example, the U.S. Department of Defense usually negotiates large procurement contracts with several contractors. Similarly, an industry-wide union (such as the United Auto Workers in the U.S.) must negotiate labor contracts with several auto manufacturers; a prosecutor may decide to negotiate plea bargaining agreements with several defendants; and a prime minister can negotiate with several parliament representatives over the allocation of pork barrel spending. This paper provides a simple framework to analyze this type of negotiations.

Specifically, this paper develops a model of multi-party negotiations in which one lead player bargains with each of several other players sequentially.[1] The key features of our

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[1]Banerji (2002) and Marshall and Merlo (2004) analyze the issue of whether a union should bargain with firms simultaneously or sequentially. See also the incomplete information model of Chatterjee and Kim (2005). Here, we assume that the bargaining games occur sequentially and under complete information.

model are that the lead player visits each other player only once, at a pre-determined order, and one-player-at-a-time. Moreover, we assume that bargaining with each player is resolved through the Nash-bargaining solution.[2]

The model has a simple and intuitive closed-form solution for the lead player's payoff. More precisely, given the parameters of the dynamic bargaining game, one can define a corresponding static lottery such that the lead player's equilibrium payoff in the ordered bargaining game is equal to the expected utility from that lottery. The corresponding static lottery is the weighted sum of independent (though not necessarily identically distributed) Bernoulli random variables. The weights are related to each player's importance from the lead player's point of view, and the probabilities of success are related to the players' bargaining power coefficients.

This key result is useful as it allows us to explain further properties of the model in a straightforward manner by invoking expected utility theory and probability theory. For example, when a buyer bargains with several potential sellers, a merger of sellers reduces (resp. increases) the buyer's equilibrium payoff if the buyer's utility function is concave (resp. convex) ceteris paribus. This follows directly from the observation that the merger makes the corresponding lottery more risky. Moreover, in the special case where the buyer's utility function is quadratic, the buyer's equilibrium payoff decreases linearly with the sellers' HHI.

For a different example, when voting is governed by a simple majority rule and a prime minister bargains with many parliament representatives over the allocation of pork barrel spending, the prime minister will be left with almost all the pork barrel budget (resp. almost no budget) if the prime minister has more (resp. less) bargaining power than the average representative. This makes use of the fact that the corresponding lottery is almost degenerate when the number of representatives is large.

The result that the bargaining game is equivalent to a simple static lottery from the perspective of the lead player implies that the order of negotiations has no impact on the equilibrium payoff of the lead player. In contrast, Marx and Shaffer (2007) and Raskovich (2007) find that the lead player may have strict preferences over the order of negotiations. Marx and Shaffer (2007) find that the order of negotiations may matter because they assume that the buyer and the first seller can write a contingent contract that extracts all the rents of the second seller.[3] In our model, we assume that contracts are non-contingent in the sense that they specify terms of trade that are fixed and do not depend on the terms of trade that will be agreed upon with future partners. Raskovich (2007) finds that the order of negotiations may matter because the buyer must incur a fixed (sunk) cost prior to visit a seller.[4] In our model, we assume zero visit costs.

[2]We should also emphasize upfront that we take the bargaining set up as given and it is not in our agenda to identify conditions under which the lead player would have a preference to structure the bargaining process in this, as opposed to some other, way. We do, however, briefly discuss this issue in section (2.2).

[3]Implicitly, the buyer and the first seller enter into an exclusive-dealing arrangement with a penalty-escape clause, as in Aghion and Bolton (1987). Marx and Shaffer (2007) show that the buyer prefers to bargain first with the seller that has the weakest bargaining position.

[4]Raskovich (2006) shows that the buyer may (inefficiently) prefer to bargain first with a seller that has a low visit cost even though the total surplus is smaller than with a seller that has a high visit cost.

The road map for the rest of the paper is as follows. Section 2 introduces the model, describes the equilibrium dynamics, and compares the model of ordered bargaining to some natural comparators from the literature. Section 3 derives the model's implications for merger analysis and outlines an application of the model to pork barrel spending. Section 4 concludes.

2 A model of ordered bargaining

There are n sellers of a good, and one buyer. Seller i has an inelastic supply of x_i units of the good.

The timing of events is as follows. The buyer bargains sequentially with each seller, in a *pre-determined order* and one at a time; when bargaining with one seller ends, the buyer approaches the next seller in the bargaining queue and so on; the buyer can approach each seller only once. Thus, we can index sellers by $i = 1, 2, ...n$, according to their position in the bargaining queue.

In particular, the buyer and each seller i engage in *Nash bargaining* over the transfer payment, t_i, that the seller requires from the buyer in exchange of his supply of the good, x_i. When the buyer bargains with seller i, their respective bargaining powers are α_i and $1-\alpha_i$. Moreover, the sellers have no outside options and, hence, their disagreement payoff is zero. The buyer does not have any outside options either, but (as we shall soon see) her disagreement payoff is not (necessarily) zero and is, in fact, determined endogenously.

Finally, we assume that the buyer and the sellers have preferences given respectively by:

$$U = u(\sum_{i=1}^{n} I_i x_i) - \sum_{i=1}^{n} I_i t_i \quad (1)$$

$$\Pi_i = I_i t_i \quad (2)$$

where $u(\cdot)$ can be any non-decreasing function and I_i is the indicator function that takes the value 1 when seller i reaches an agreement with the buyer. This completes the description of the model. Below, we illustrate how the model plays out in the context of a two-seller example.

Example 1 *There are two sellers, 1 and 2, with supplies of x_1 and x_2 units of the good, respectively. The buyer bargains first with seller 1 and then with seller 2. The bargaining power of both sellers is $\frac{1}{2}$ and the buyer's utility function is given by a non-decreasing function, $u(\cdot)$, with $u(0) = 0$ and $u(x_1 + x_2) = 1$.*

We proceed by backwards induction. We first analyze how bargaining with seller 2 will play out. We distinguish between two cases, depending on whether the buyer reached an agreement with seller 1 in the previous round.

- ***Bargaining with seller 2***

  Case 1: *The buyer did not reach an agreement with seller 1.*

- *The buyer's disagreement payoff is* $u(0) = 0$.
- *The seller's marginal contribution is* $u(x_2)$.
- *The buyer and seller 2 split* $u(x_2)$ *fifty-fifty and walk away with* $\frac{u(x_2)}{2}$ *each.*

Case 2: *The buyer reached an agreement with seller 1 for a transfer* t_1.

- *The buyer's disagreement payoff is* $u(x_1) - t_1$.
- *The seller's marginal contribution is* $u(x_1 + x_2) - u(x_1) = 1 - u(x_1)$.
- *The buyer and seller 2 split the marginal contribution of seller 2 fifty-fifty.*
- *The buyer walks away with* $(u(x_1) - t_1) + \frac{1-u(x_1)}{2} = \frac{1+u(x_1)}{2} - t_1$.
- *Seller 2 walks away with* $\frac{1-u(x_1)}{2}$.

Next, we look at how bargaining with seller 1 will play out.

- ***Bargaining with seller 1***
  - *The buyer's disagreement payoff is* $\frac{u(x_2)}{2}$, *from case 1 above.*
  - *If the buyer and seller 1 agree on a transfer payment* t_1, *then the buyer's payoff will be* $\frac{1+u(x_1)}{2} - t_1$, *from case 2 above.*
  - *The bargaining surplus is the difference between* $\frac{1+u(x_1)}{2}$ *and* $\frac{u(x_2)}{2}$. *This is equal to* $\frac{1+u(x_1)-u(x_2)}{2}$, *which the buyer and the seller split fifty-fifty.*
  - *The buyer walks away with* $\frac{1+u(x_1)}{2} - \frac{1}{2}\frac{1+u(x_1)-u(x_2)}{2} = \frac{1+u(x_1)+u(x_2)}{4}$.
  - *Seller 1 walks away with* $\frac{1}{2}\frac{1+u(x_1)-u(x_2)}{2} = \frac{1+u(x_1)-u(x_2)}{4}$.
  - *Seller 2 walks away with* $\frac{1-u(x_1)}{2}$.

For future reference, notice that the order of bargaining does not affect the buyer's payoff. It does affect, however, the payoffs of the sellers. (To see these, swap x_1 *and* x_2 *in the last three expressions.)* ■

Next, we discuss the solution to the general model with multiple sellers and arbitrary bargaining power coefficients.

2.1 Bargaining dynamics and equilibrium

The model can be solved by backwards induction as follows. Suppose that prior to visiting seller n, the last seller in the bargaining queue, the buyer has reached agreements with sellers whose total supply of the good sums up to S (for *state*).[5] Then, according to the Nash bargaining solution, the buyer and seller n will agree on the transfer payment, t_n, that solves the following maximization problem:[6]

[5] Notice that the marginal contribution of seller n's supply to the buyer does not depend on any prior transfer payments. Hence, S, the total supply that the buyer has accumulated so far, is the only relevant state variable. This is a consequence of the *quasi-linearity* of the buyer's preferences.

[6] See, for example, Osborne and Rubinstein (1990), chapter 2, section 5.

$$\max_{t_n}[u(S+x_n)-t_n-u(S)]^{\alpha_n}t_n^{1-\alpha_n} \tag{3}$$
$$\Rightarrow t_n^*=(1-\alpha_n)[u(S+x_n)-u(S)] \tag{4}$$

The first term in equation (3) is the difference in the buyer's payoff from reaching an agreement ($u(S+x_n)-t_n$) and not reaching an agreement ($u(S)$), raised to the buyer's bargaining power (α_n). Similarly, the second term is the difference in the seller's payoff from reaching (t_n) and not reaching (zero) an agreement, raised to the seller's bargaining power (1-α_n). Seller n walks away with t_n^* from equation (4) above, whereas the buyer walks away with:

$$u(S+x_n)-t_n^*=\alpha_n u(S+x_n)+(1-\alpha_n)u(S) \tag{5}$$

Suppose now that the buyer is about to embark on bargaining with seller i, having already secured a total of S units of the good from sellers earlier in the queue, i.e., from sellers $1,2,..,i-1$. Denote by $V(S,i)$ the payoff that the buyer will eventually obtain, after having dealt with all sellers in the bargaining queue, *not including any past payments* to sellers $1,2,...,i-1$. From equation (5), we have:

$$V(S,n)=\alpha_n u(S+x_n)+(1-\alpha_n)u(S) \tag{6}$$

If the buyer reaches an agreement with seller i, for a transfer payment of t_i, then the buyer will eventually obtain a payoff of $V(S+x_i,i+1)-t_i$ (not including any past payments). Likewise, if the buyer does not reach an agreement with seller i, then the buyer will eventually obtain a payoff of $V(S,i+1)$. Notice that the expression $V(S,i+1)$ is the buyer's disagreement payoff and that it is determined endogenously. Then, according to the Nash bargaining solution, the buyer and seller i will agree on the transfer payment, t_i, that solves the following maximization problem:

$$\max_{t_i}[(V(S+x_i,i+1)-t_i)-V(S,i+1)]^{\alpha_i}t_i^{1-\alpha_i} \tag{7}$$
$$\Rightarrow t_i^*=1-\alpha_i)[V(S+x_i,i+1)-V(S,i+1)] \tag{8}$$

Seller i walks away with t_i^*, whereas the buyer walks away with:

$$V(S+x_i,i+1)-t_i^*=\alpha_i V(S+x_i,i+1)+(1-\alpha_i)V(S,i+1) \tag{9}$$

Thus, one can obtain the following recursive relationship for the buyer's payoff, as a function of the state:

$$V(S,i)=\alpha_i V(S+x_i,i+1)+(1-\alpha_i)V(S,i+1) \tag{10}$$

which implies that the buyer's *equilibrium payoff* is $V(0,1)$.

We now show that the buyer's payoff equals the expected utility of a weighted sum of independent Bernoulli random variables.

Proposition 1 *Under the Nash bargaining solution, the buyer's payoff is given by*

$$V(S,i) = E[u(S + \sum_{j=i}^{n} y_j x_j)] \tag{11}$$

where each y_j is a Bernoulli random variable that takes the value 1 with probability α_j, i.e., $y_j \sim Ber(\alpha_j)$.

Proof. The proof follows an inductive argument. We first look at the base case $i = n$. We can re-write equation (6) as:

$$V(S,n) = E[u(S + y_n x_n)] \tag{12}$$

where $y_n \sim Ber(\alpha_n)$. Hence, the claim in equation (11) is true for the case $i = n$.

Suppose now that the result is true for $i = k > 1$. Then, we can write:

$$V(S,k) = E[u(S + \sum_{j=k}^{n} y_j x_j)] \tag{13}$$

where $y_j \sim Ber(\alpha_j)$, for all $j \geq k$.

We now show that the result is true for the case $i = k-1$. Using equations (10) and (13), we have:

$$V(S,k-1) = \alpha_{k-1} E[u(S + x_{k-1} + \sum_{j=k}^{n} y_j x_j)] + (1-\alpha_{k-1}) E[u(S + \sum_{j=k}^{n} y_j x_j)] \tag{14}$$

where $y_j \sim Ber(\alpha_j)$, for all $j \geq k$. It then follows that:

$$V(S,k-1) = E[u(S + y_{k-1}x_{k-1} + \sum_{j=k}^{n} y_j x_j)] = E[u(S + \sum_{j=k-1}^{n} y_j x_j)] \tag{15}$$

where $y_j \sim Ber(\alpha_j)$, for all $j \geq k-1$. ■

Two comments are in order. First, proposition 1 allows us to give to the buyer's equilibrium payoff the following intuitive interpretation. Suppose that, instead of bargaining with the buyer, each seller i were to flip a coin that lands on HEADS with probability $1-\alpha_i$. (Notice that $1-\alpha_i$ is seller i's bargaining power when bargaining with the buyer.) If the coin lands on HEADS, then the seller and the buyer do not trade; otherwise the seller surrenders his supply of x_i units to the buyer *for free*. Then, according to proposition 1, the buyer's expected utility from this lottery is equal to the buyer's equilibrium payoff in the original bargaining game. A direct consequence of this equivalence is that the order of bargaining does not affect the buyer's equilibrium payoff. We state this as a corollary below.

Corollary 1 *The order of bargaining does not affect the buyer's equilibrium payoff.*

Second, by reducing the buyer's payoff to the weighted sum of independent Bernoulli random variables, proposition 1 allows us to use tools from expected utility theory and probability theory to flesh out other properties of the model. The following sections will demonstrate that further intuition and several results can be obtained in a straightforward manner by using proposition 1 in conjunction with standard results from probability and expected utility theory.

Next, we compare the model of ordered bargaining to some natural comparators.

2.2 Ordered bargaining vs. other natural benchmarks

We will compare the model of ordered bargaining to three natural comparators: the competitive equilibrium, simultaneous Nash bargaining and the Shapley value. Our objective is neither to provide a normative analysis of which bargaining set up the buyer would prefer, nor to contrast the model of ordered bargaining against every conceivable alternative in the literature. Rather, by comparing the model of ordered bargaining to some natural comparators, we would like to convey additional intuition behind our model. In the process, we will also provide a geometric interpretation and a numerical example.

To facilitate the comparison across these models, we will restrict attention to the symmetric case, where there are n identical sellers, each with bargaining power $1-\alpha$ and a supply of $\frac{1}{n}$ units. We will also normalize the utility function $u(\cdot)$, so that $u(0) = 0$ and $u(1) = 1$.

2.2.1 Ordered bargaining vs. the competitive equilibrium

Under the competitive equilibrium, there exists a per-unit price, p, that the buyer must pay to all sellers. Given this per-unit price p, the buyer forms a demand for the good, call it q. This is derived by maximizing the buyer's payoff, as shown below:

$$\begin{aligned} \max_q\ & u(q) - pq \\ \Rightarrow\ & u'(q) = p \end{aligned} \tag{16}$$

where $u(\cdot)$ is a non-decreasing concave function.[7]

In equilibrium it must be that the price p is such that the buyer's demand at that price equals the total supply (normalized to 1 unit). Hence, the equilibrium price and buyer's payoff are given by:

[7] With a convex utility function the definition of the competitive equilibrium is problematic. Notice, however, that in the ordered bargaining model the only restriction on the utility function is that it must be non-decreasing.

$$p = u'(1) \tag{17}$$
$$U = u(1) - u'(1) = 1 - u'(1) \tag{18}$$

When there are *many* identical sellers, the competitive equilibrium has an alternative interpretation: it is as if the buyer bargains with all n sellers simultaneously and pays to each of them the full extent of their marginal contribution, i.e., pays $u'(1)$ to each seller. In contrast, under ordered bargaining: a) different sellers have different marginal contributions depending on their position in the queue, and b) the buyer and each seller split that seller's marginal contribution according to their relative bargaining power.

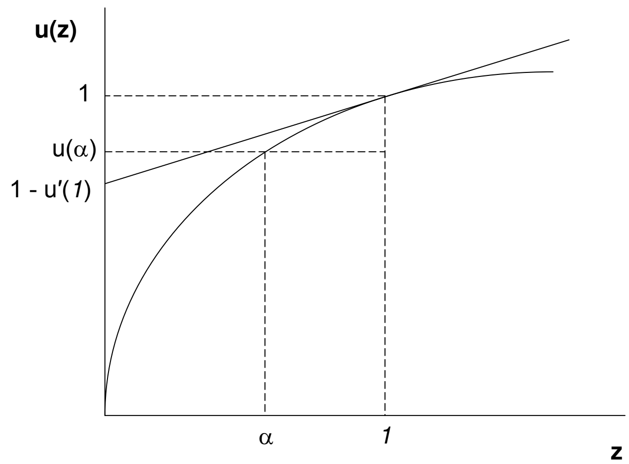

Figure 1. Ordered bargaining vs. the competitive equilibrium

Figure 1 gives a geometric interpretation of the buyer's payoff under the two models. Under the competitive equilibrium, the buyer's payoff is the distance between $u(1) = 1$ and the point where the tangent at $u(1)$ meets the vertical axis, i.e., $1 - u'(1)$. Invoking proposition 1 and the law of large numbers, one can see that under ordered bargaining with many sellers the buyer's equilibrium payoff is approximately equal to $u(\alpha)$. It also follows from the figure that as the utility function, $u(\cdot)$, becomes less concave, the threshold bargaining power coefficient that makes the buyer's payoff equal under both models decreases. In the extreme case where the utility function is linear, the buyer is better off under ordered bargaining for any bargaining power strictly greater than zero.

Below, we illustrate these ideas in the context of a stylized example.

Example 2 *Let the utility function,* $u(\cdot)$*, be of the CARA family, normalized so that* $u(0) = 0$ *and* $u(1) = 1$*:*

$$u(z) = \frac{1 - e^{-\theta z}}{1 - e^{-\theta}} \tag{19}$$

where $\theta > 0$. Then, the buyer's payoff under the competitive equilibrium is equal to:

$$u(1) - u'(1) = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - e^{-\theta}} \tag{20}$$

whereas under ordered bargaining, the buyer's equilibrium payoff is equal to:

$$\begin{aligned} E[\frac{1 - e^{-\theta \frac{1}{n}\sum_{i=1}^{n} y_i}}{1 - e^{-\theta}}] &= \frac{1}{1 - e^{-\theta}}(1 - E[e^{-\frac{\theta}{n}\sum_{i=1}^{n} y_i}]) \\ = \frac{1}{1 - e^{-\theta}}(1 - E[\prod_{i=1}^{n} e^{-\frac{\theta}{n} y_i}]) &= \frac{1}{1 - e^{-\theta}}(1 - \prod_{i=1}^{n} E[e^{-\frac{\theta}{n} y_i}]) \\ &= \frac{1}{1 - e^{-\theta}}(1 - (1 - \alpha + \alpha e^{-\frac{\theta}{n}})^n) \end{aligned}$$

where $y_i \sim Ber(\alpha)$. As the number of sellers, n, goes to infinity, the buyer's payoff under ordered bargaining converges to:

$$\frac{1 - e^{-\theta\alpha}}{1 - e^{-\theta}} \tag{21}$$

Therefore, by comparing the expressions (20) and (21) one can see that the threshold bargaining power, α^*, that makes the buyer's payoff equal under both models is given by:

$$\begin{aligned} 1 - e^{-\theta\alpha^*} &= 1 - e^{-\theta} - \theta e^{-\theta} \\ \Rightarrow \alpha^* &= \frac{\theta - \ln(1+\theta)}{\theta} \end{aligned} \tag{22}$$

It is straightforward to show that the expression $\frac{\theta - \ln(1+\theta)}{\theta}$ is increasing in θ. In other words, as the utility function becomes less concave (θ decreases), the threshold bargaining power, α^*, decreases. One can also verify that in the limit, as θ goes to zero (i.e., the utility function becomes linear), the threshold bargaining power, α^*, goes to zero. ■

2.2.2 Ordered bargaining vs. simultaneous Nash bargaining

We now consider the case where the buyer Nash-bargains with each seller simultaneously, but separately with each seller.[8] We look for a symmetric equilibrium of this game as follows. Suppose that the buyer reaches an agreement with all sellers, but seller i, for a transfer payment of t^*_{-i}. Then, the Nash bargaining solution dictates that the buyer and seller i will reach an agreement for the transfer payment t_i that maximizes the following expression:

[8]Note that this is different from the $n-$player Nash bargaining set up.

$$\underset{t_i}{max}\ \{[1-(n-1)t^*_{-i}-t_i]-[u(\frac{n-1}{n})-(n-1)t^*_{-i}]\}^{\alpha}\{t_i\}^{1-\alpha} \tag{23}$$

where the terms in the brackets correspond to the buyer's agreement and disagreement payoffs, respectively, when bargaining with seller i. The Nash bargaining solution is given by:

$$t_i = [1-\alpha][1-u(\frac{n-1}{n})] \tag{24}$$

In other words, the buyer concedes to each seller the fraction $1-\alpha$ of each seller's marginal contribution. Hence, the buyer's equilibrium payoff under simultaneous Nash bargaining is equal to:

$$1-[1-\alpha][1-u(\frac{n-1}{n})]n \tag{25}$$

We can also state the following result.

Proposition 2 *For any (strictly) concave utility function, $u(\cdot)$, the buyer's equilibrium payoff is (strictly) lower under ordered bargaining than under simultaneous Nash bargaining.*

Proof.

We distinguish between the following three cases:

Case 1: $n = \frac{1}{1-\alpha}$

Let the number of sellers, n, be equal to $\frac{1}{1-\alpha}$. Then, it follows from equation (25) that the buyer's payoff under simultaneous Nash bargaining is equal to $u(\alpha)$. Moreover, as established by proposition 1, the buyer's equilibrium payoff under ordered bargaining is equal to the expected utility of a lottery with expected value equal to α. Hence, for any concave utility function, $u(\alpha)$ is an upper bound for the buyer's payoff under ordered bargaining.

Case 2: $n > \frac{1}{1-\alpha}$

Having established that the required result is true for $n = \frac{1}{1-\alpha}$, it suffices to show that buyer's payoff under simultaneous Nash bargaining is increasing in the number of sellers, n. Differentiating the buyer's payoff under simultaneous Nash bargaining, given in equation (25), yields:

$$(1-\alpha)\{u'(\frac{n-1}{n})\frac{1}{n}-[1-u(\frac{n-1}{n})]\} > 0 \tag{26}$$

The term $u'(\frac{n-1}{n})\frac{1}{n}$, is an *approximation* of each seller's marginal contribution, whereas the term $[1-u(\frac{n-1}{n})]$ is the *actual* marginal contribution. The inequality follows once one observes that for any concave utility function this approximation overstates the actual marginal contribution.

Case 3: $n < \frac{1}{1-\alpha}$

We can re-write the buyer's payoff under simultaneous Nash bargaining, given in equation (25), as:

$$1 \times [1 - n(1-\alpha)] + u(\frac{n-1}{n}) \times n(1-\alpha) \tag{27}$$

Note that the term $n(1-\alpha)$ is between 0 and 1, so that the buyer's payoff under simultaneous Nash bargaining can be thought of as the expected utility of the lottery that pays 1 with probability $[1-n(1-\alpha)]$, and $\frac{n-1}{n}$ otherwise. We will denote this lottery as L_{SB}, noting that its expected value is equal to α.

We will show that the lottery that describes the buyer's payoff under ordered bargaining (see proposition 1) is a mean preserving spread of the lottery L_{SB}. Given that the two lotteries have equal means (α) and that the lottery L_{SB} is a two-outcome lottery, it suffices to show that the lottery L_{SB} puts less weight at the extreme point 1.

Under the lottery L_{SB} the probability of the outcome 1 is equal to $1 - n(1-\alpha)$. In contrast, under the lottery that describes the buyer's payoff under ordered bargaining the probability of the outcome 1 is equal to $\alpha^n \geq 1 - n(1-\alpha)$. The inequality follows once we re-write it as:

$$\alpha^n \geq 1 - n(1-\alpha) \Leftrightarrow n \geq \frac{1-\alpha^n}{1-\alpha} \Leftrightarrow n \geq 1 + \alpha + \alpha^2 + ... + \alpha^{n-1} \tag{28}$$

The last sum involves n terms, each being less than 1, so the sum can be at most equal to n. ■

The reverse will be true for convex utility functions. This suggests that in negotiations where the institutional context does not by itself dictate the type of bargaining process, the lead negotiator will seek to structure the bargaining process sequentially or simultaneously, depending on the convexity or concavity of her payoff function.

2.2.3 Ordered bargaining vs. the Shapley value

If the buyer and the sellers have equal bargaining power, i.e., $\alpha = \frac{1}{2}$, a natural way to cast the motivating buyer-seller problem in the context of cooperative game theory with transferable utility is to define the following characteristic function:

$$v(S) = \begin{cases} u(\frac{|S|-1}{n}) & \text{, if the buyer is a member of coalition } S \\ 0 & \text{, otherwise} \end{cases} \tag{29}$$

In this game, the Shapley value for the buyer is equal to the expected utility of the following lottery:

$$L_{Shapley} = \begin{cases} 0, \text{ with probability } \frac{1}{n+1} \\ \frac{1}{n}, \text{ with probability } \frac{1}{n+1} \\ \frac{2}{n}, \text{ with probability } \frac{1}{n+1} \\ ... \\ 1, \text{ with probability } \frac{1}{n+1} \end{cases} \tag{30}$$

To see this, note that the Shapley value can be thought of as a player's expected marginal contribution, under a scenario where players join the grand coalition of all players sequentially, one-at-a-time and at random. Under the characteristic function defined above, the buyer's marginal contribution can be $0, u(\frac{1}{n}), u(\frac{2}{n}), ..., u(1)$, depending on whether the buyer is chosen to join the grand coalition, first, second, third, etc, each of which scenarios is equally likely. In contrast, under ordered bargaining proposition 1 allows us to characterize the buyer's payoff as the expected utility of the following lottery:

$$L_{OB} = \begin{cases} 0, \text{ with probability } (\frac{1}{2})^n C(n,0) \\ \frac{1}{n}, \text{ with probability } (\frac{1}{2})^n C(n,1) \\ \frac{2}{n}, \text{ with probability } (\frac{1}{2})^n C(n,2) \\ ... \\ 1, \text{ with probability } (\frac{1}{2})^n C(n,n) \end{cases} \quad (31)$$

For example, when there are two sellers ($n = 2$), the Shapley value for the buyer is equal to the expected utility of the lottery that pays 0, 1 or 2, with equal probability, whereas under ordered bargaining the buyer's equilibrium payoff is equal to the expected utility of the lottery that pays 0, 1, or 2, with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively.

It is immediate to see that the lottery $L_{Shapley}$ is a mean preserving spread of the lottery L_{OB}. This observation allows us to state the following result.

Proposition 3 *For any concave utility function $u(\cdot)$, the buyer's equilibrium payoff under ordered bargaining is higher than the payoff attributed to the buyer under the Shapley value. The opposite will be true for any convex utility function.*

3 Applications

In this section we will illustrate how the model can be fruitfully applied to various applications. We will first discuss one application in industrial organization, in the context of merger analysis in particular, and we will then outline a political economy application.

3.1 Implications for merger analysis

Initially, the buyer bargains with n sellers. We would like to know how the buyer's payoff will change if two of these sellers merge. We assume that initially all sellers have the same bargaining power, $1 - \alpha$, which will also be the bargaining power of all sellers (including the merged party) post-merger. Without loss of generality, we let the merging parties by sellers 1 and 2. (Recall that the order of bargaining does not matter for the buyer's payoff.)

Invoking proposition 1, the buyer's equilibrium payoff under ordered bargaining equals the expected utility of a weighted sum of Bernoulli random variables. We denote by L_{pre} the corresponding pre-merger weighted sum (or lottery) given by:

$$L_{pre} = \sum_{i=1}^{n} y_i x_i \tag{32}$$

where $y_i \sim Ber(\alpha)$.

After the merger the buyer's equilibrium payoff will be the expected utility of the corresponding post-merger weighted sum (or lottery), L_{post}, given by:

$$L_{post} = y_{12}(x_1 + x_2) + \sum_{i=3}^{n} y_i x_i \tag{33}$$

where $y_{12}, y_i \sim Ber(\alpha)$.

Denote by F_{pre} and the F_{post} the cumulative distribution functions for the corresponding pre and post-merger lotteries. Then, we can state the following result.

Lemma *The distribution of the corresponding post-merger lottery, F_{post}, is a mean-preserving spread of the distribution of the corresponding pre-merger lottery, F_{pre}.*

Then, the next result follows from Rothschild and Stiglitz (1970).

Proposition 4 *Suppose that any two sellers merge. Then, if the buyer's utility function $u(\cdot)$, is concave (resp. convex), a merger decreases (resp. increases) the buyer's equilibrium payoff.*

This result is reminiscent of a standard result from oligopoly theory. In Cournot and Bertrand models of oligopolistic competition a merger harms or benefits consumers, depending on whether the merging parties produce goods that are substitutes or complements, respectively. To see the connection between this standard result and proposition 4, notice that when the buyer's utility function is convex, the supplies of any two producers can be thought of as complements. Similarly, when the buyer's utility function is concave, the supplies of any two producers can bethought of as substitutes. It should be pointed out, however, that our result is derived in the context of an entirely different trading mechanism and driven by different forces. Hence, we view this connection between proposition 4 and the standard result from Cournot/Bertrand models as only a superficial one.

Below, we provide an illustrative example which describes some sharper results that can be obtained when the buyer's utility function admits a mean-variance representation. In particular, we show that when the buyer's utility function is quadratic (and all sellers have the same bargaining power), the buyer's payoff is *linear* in the Herfindahl-Hirschman index.

Example 3 *Initially there are n sellers with bargaining power $1 - \alpha$ each. We normalize each seller's supply, x_i, so that $\sum_{i=1}^{n} x_i = 1$, and we assume that the buyer's utility function is quadratic:*

$$u(z) = z - \frac{1}{2} z^2 \tag{34}$$

Then, the buyer's pre-merger equilibrium payoff, U^{pre}, equals:

$$U^{pre} = E[u(L_{pre})] = E[L_{pre}] - \frac{1}{2}E[L_{pre}^2] \quad (35)$$
$$= E[L_{pre}] - \frac{1}{2}(E[L_{pre}]^2 + Var[L_{pre}])$$
$$= E[L_{pre}] - \frac{1}{2}E[L_{pre}]^2 - \frac{1}{2}Var[L_{pre}]$$

where the lottery L_{pre} is defined in equation (32). Substituting $E[L_{pre}] = \alpha$ and $Var[L_{pre}] = \alpha(1-\alpha)\sum_{i=1}^{n} x_i^2$ we obtain the following closed-form expression for the buyer's pre-merger payoff:

$$U^{pre} = \alpha - \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha(1-\alpha)\sum_{i=1}^{n} x_i^2 \quad (36)$$
$$= \alpha(1 - \frac{1}{2}\alpha) - \frac{1}{2}\alpha(1-\alpha)HHI_{pre}$$

where HHI_{pre} denotes the pre-merger Herfindahl-Hirschman index. In other words, when the buyer's utility function is quadratic and all sellers have the same bargaining power, then buyer's payoff is linear *in the HHI.*

Suppose now that sellers 1 and 2 merge. Moreover, assume that all sellers (including the merging party) maintain their pre-merger bargaining power, $1-\alpha$. Then, mimicking our earlier steps, we can see that the buyer's post-merger equilibrium payoff, U^{post}, is linear in the post-merger HHI index, HHI_{post}, as shown below:

$$U^{post} = E[u(L_{post})] = E[L_{post}] - \frac{1}{2}E[L_{post}]^2 - \frac{1}{2}Var[L_{post}] \quad (37)$$
$$= \alpha(1 - \frac{1}{2}\alpha) - \frac{1}{2}\alpha(1-\alpha)HHI_{post}$$

where the lottery L_{post} is defined in equation (33). Subtracting the buyer's post-merger payoff, U^{post}, from her pre-merger payoff, U^{pre}, shows that the merger decreases the buyer's payoff by:

$$U^{pre} - U^{post} = \frac{1}{2}\alpha(1-\alpha)[HHI_{post} - HHI_{pre}] = \alpha(1-\alpha)x_1x_2 \quad (38)$$

3.2 The allocation of pork barrel spending

Next, we suggest a potential application of the model to the allocation of pork barrel spending.

There is a Prime Minister and a parliament of n representatives. The Prime Minister is asking from parliament to pass a bill that will give her the authority to access \$1 from the budget and spend it at her discretion. Each representative can cast only one vote and the bill requires at least a fraction m of the votes in order to pass.

The Prime Minister bargains sequentially with each representative in a pre-determined order, as before. We index the representatives by $i = 1, 2, ..., n$, according to their position in the bargaining queue. The Prime Minister and each representative i bargain over what fraction of the dollar will be allocated for funding projects in the representative's district, denoted by t_i, should the representative vote in favor of the bill and the bill passes. The Prime Minister can use any money left over to finance her own pet causes. The preferences for the Prime Minister and each representative i are given, respectively, by:

$$U = u(\sum_{i=1}^{n} I_i) - \sum_{i=1}^{n} I_i t_i \tag{39}$$
$$\text{where } u(x) = \begin{cases} 1, & \text{if } x \geq mn \\ 0, & \text{otherwise} \end{cases}$$
$$\Pi_i = I_i t_i \tag{40}$$

where I_i is the indicator function that takes the value 1 when representative i agrees to vote in favor of the bill.

Suppose that the Prime Minister's bargaining power when bargaining with each representative is α. Then, invoking the result in proposition 1, we know that the equilibrium payoff to the Prime Minister under ordered bargaining equals the expectation of the sum of n i.i.d. Bernoulli random variables, each with mean α. Moreover, using the fact that this sum follows a binomial distribution with parameters n and α, we can express the Prime Minister's equilibrium payoff as:

$$U = \sum_{j=M}^{n} \frac{n!}{j!(n-j)!} \alpha^j (1-\alpha)^{n-j} \tag{41}$$

where M is the smallest integer such that $M \geq mn$. For example, suppose that there are $n = 10$ representatives, the Prime Minister's bargaining power is $\frac{1}{2}$ and a majority of $m = 0.6$ is needed, i.e. $M = 6$. Then, the equilibrium payoff to the Prime Minister is approximately 0.38. In other words, the Prime Minister would have to concede 62% of the dollar to the first six representatives in the queue and will keep the remaining 38% for her own pork barrel spending.

Furthermore, when the number of representatives, n, is large, we can use the normal approximation of the binomial distribution to obtain the following result.

Proposition 5 *As the number of representatives n increases, the Prime Minister's equilibrium payoff converges to 1, 0 or $\frac{1}{2}$, depending on whether the required majority, m, is smaller, larger or equal to the Prime Minister's bargaining power α, respectively.*

Proof. The Prime Minister's equilibrium payoff equals the probability $Pr[Z \geq nm]$, where Z follows a binomial distribution with parameters n and α. As n increases, the binomial distribution with parameters n and α approaches the normal distribution with mean $n\alpha$ and variance $n\alpha(1-\alpha)$. Hence, the Prime Minister's equilibrium payoff is approximately equal to

$$Pr[Z \geq nm] = Pr[\frac{Z - n\alpha}{\sqrt{n\alpha(1-\alpha)}} \geq \frac{nm - n\alpha}{\sqrt{n\alpha(1-\alpha)}}] \approx 1 - \Phi(\frac{n(m-\alpha) - \frac{1}{2}}{\sqrt{n\alpha(1-\alpha)}}) \quad (42)$$

where $\Phi(\cdot)$ denotes the cumulative density of the standard normal distribution. (The fraction $-\frac{1}{2}$ in the second line of equation (42) is the continuity correction.) It is straightforward to see that as n goes to infinity, the term $\frac{n(m-\alpha)-\frac{1}{2}}{\sqrt{n\alpha(1-\alpha)}}$ approaches ∞, $-\infty$ or 0, depending on whether $m > \alpha$, $m < \alpha$ or $m = \alpha$, respectively. The result follows once one notices that $\Phi(\infty) = 1$, $\Phi(-\infty) = 0$ and $\Phi(0) = \frac{1}{2}$. ■

In other words, when the Prime Minister's bargaining power is greater than the required majority, i.e., $\alpha > m$, she will have a preference for very large committees (and vice versa). This result has the following implication for committee design. Suppose that decisions are made by simple majority (i.e., $m = \frac{1}{2}$, as is often the case) and that the Prime Minister has greater bargaining power than the representatives (as is most likely to be the case). Then, to discourage the Prime Minister from engaging in pork barrel spending initiatives, budget allocation decisions should be assigned to small committees.

4 Concluding remarks

We presented a model of ordered bargaining between a buyer and several sellers, with Nash bargaining at each stage. We showed that the model has the property that the buyer's equilibrium payoff is equal to a weighted sum of independent Bernoulli random variables. In turn, this property allowed us to uncover further properties of the model in an easy and straightforward manner by invoking well-known results from expected utility and probability theory. We discussed these properties in the context of merger analysis and the political economy of pork barrel spending, illustrating that the model provides a useful framework to analyze business and political interactions where a lead negotiator bargains with several partners or allies.

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